Predicate logic

1. Why propositional logic is not enough?

- **Discussion:** (i) Does (1a) contradict (1b)? [Two sentences are contradictory iff they cannot be simultaneously true.] (ii) If it does, can you show this contradiction by propositional logic?

(1) a. Mary is wearing a blue skirt.
    b. Nobody is wearing a blue skirt.

- **Discussion:** How did we translate the following quantificational sentences in set-theoretic notations?

(2) a. Kitty is a cat.
    b. Some cat meows.
    c. Every cat meows.

2. Vocabulary and syntax of predicate Logic

- **Vocabulary of predicate logic**

(3) a. Individual constants: $d, n, j, m, ...$
    b. Individual variables: $x, y, z, ...$
       (Individual variables and constants together are called *terms*.)
    c. Predicates: $P, Q, R, ...$, each with a fixed finite number of argument places.
    d. Connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
    e. Quantifiers:
       i. existential quantifier $\exists$ (means 'some' in the sense of 'at least one, possibly more'),
       ii. universal quantifier $\forall$ (means 'every')
    f. Constituency labels: ( ), [ ], commas

- **Syntax of predicate logic**

(4) Basic expressions
  a. terms
     – individual constants: d, n, j, m, ...
     – individual variables: x, y, z, ...
  b. predicates:
     – unary predicates: happy, bored, linguist, philosopher, cellist, smokes
     – binary predicates: kissed, loves, admires
     – n-ary predicates: ??

(5) Well-formed formulas
  a. Every sentence variable is a wff.

b. If t_1 and t_1 are terms, then $t_1 = t_2$ is a wff.
c. If t_1 is an individual term and P is a one-place predicate term, then $P(t_1)$ is a wff.
d. If t_1 and t_2 are individual terms and P is a two-place predicate term, then $P(t_1, t_2)$ is a wff.
e. If P is a n-place predicate and $t_1, t_2, ..., t_n$ are terms, then $P(t_1, t_2, ..., t_n)$ is a wff.
f. If α is a wff and x is an individual variable, then $[\exists x\alpha]$ and $[\forall x\alpha]$ are wffs.[1]
g. If α and β are wffs, then $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$ are wffs.
h. Nothing else is a wff of predicate logic.

We can also drop some of the brackets like what we did in propositional logic.

Exercise: Following the recursive rules above, identify whether each of the following strings is a wff of predicate logic. (H is a one-place predicate, L is a two-place predicate, jd are individual constants, xy are individual variables.)

(6) a. H
b. j
c. $j = d$
d. $H(j)$
e. $L(j)$
f. $L(j, x)$
g. $\forall x H(y)$
h. $\forall x \exists y[H(x) \to L(d, x)]$

- **Scope, bound/free variables, closed/open formula**
  - In $\exists x\alpha$ and $\forall x\alpha$, α is the *scope* of $\exists x$ and $\forall x$.

(7) a. $\forall x\underline{\exists y\underline{[H(x) \to L(d, x)]}}$ (scope of $\forall x$) (scope of $\exists y$)
b. $\exists x\underline{[Q(x) \wedge \forall y\underline{[P(y) \to \exists z\underline{S(x, y, z)}]}]}$ (scope of $\exists x$) (scope of $\forall y$) (scope of $\exists z$)

  - Bound/free variables, closed/open formula:
    * An occurrence of a variable x is *bound* if it occurs in the scope of $\exists x/\forall x$, otherwise it is *free*.

(8) a. $\exists x[P(x)]$, $\forall x[P(y) \to Q(x)]$
b. $P(x)$, $\forall y P(x)$

    * Every occurrence of a variable x can only be bound at most once.
    Example: In (9), the occurrence of x in $Q(x)$ has been bound by its closest eligible binder $\exists x$, and thus is not bound by $\forall x$.

(9) $\forall x[P(x) \to \exists x Q(x)]$ is equivalent to $\forall x[P(x) \to \exists y Q(y)]$

    * A wff is *closed* iff it does not contain any free variables, otherwise it is *open.*

[1] This definition allows vacuous quantification (e.g., $\exists x P(j)$) and is more permissive than the one in Allwood et al. (1977).

3. Semantics of predicate logic

3.1. Without variables

- **Models**

  The truth value of any statement in predicate logic depends on the domain of discourse and the choice of semantic values for the constants and predicates. We interpret expressions of predicate logic in **models**. A model M is a pair $\langle D, I \rangle$, in particular:

  – D is the **domain of discourse** (i.e., the set of considered individuals).

  – I is an **interpretation function** that assigns a semantic value to each basic non-logical constant expression.[2]

  For an expression α, $[\![\alpha]\!]^M$ is called the *interpretation/denotation* of α relative to M.

– Example: a toy language L

Categories	Expressions in L	NL counterparts
Names	a, b, c, d	Andy, Billy, Cindy, Danny
1-place predicates	H, C	happy, cried
2-place predicates	L, K	dislike, know

(10) $M_1 = \langle D_1, I_1 \rangle$, where

a. D_1 = {Andy, Billy, Cindy, Danny}

b. Interpreting names/ individual constants

$I_1(a)$ = Andy, $I_1(b)$ = Billy, $I_1(c)$ = Cindy, $I_1(d)$ = Danny

c. Interpreting predicates

$I_1(H)$ = {Andy, Billy}

$I_1(C)$ = {Andy, Billy, Cindy}

$I_1(L)$ = {⟨Andy, Danny⟩, ⟨Billy, Cindy⟩}

$I_1(K)$ = {⟨Andy, Danny⟩, ⟨Billy, Cindy⟩, ⟨Cindy, Andy⟩}

- **Semantics of predicate logic (without variables)**

  (a and b are individual constants, P is a predicate, ϕ and ψ are formulas.)

(11) **Semantics of basic expressions**

a. $[\![a]\!]^M = I(a)$

b. $[\![P]\!]^M = I(P)$

(12) **Semantics of formulas**

a. $[\![a = b]\!]^M = 1$ iff $[\![a]\!]^M = [\![b]\!]^M$

b. $[\![P(a)]\!]^M = 1$ iff $[\![a]\!]^M \in [\![P]\!]^M$.

c. $[\![P(a_1, a_2, ..., a_n)]\!]^M = 1$ iff $\langle [\![a_1]\!]^M, [\![a_2]\!]^M, ..., [\![a_n]\!]^M \rangle \in [\![P]\!]^M$.

d. $[\![\neg\phi]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

e. $[\![\phi \wedge \psi]\!]^M = 1$ iff $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

f. $[\![\phi \vee \psi]\!]^M = 1$ iff $[\![\phi]\!]^M = 1$ or $[\![\psi]\!]^M = 1$.

[2] The value of a basic expression depends on the model, while the contribution of a connective to the meaning is the same regardless of the model. Hence, predicates and individual constants are *non-logical constants*, while connectives are *logical constants*.

g. $[\![\phi \rightarrow \psi]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.
h. $[\![\phi \leftrightarrow \psi]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

Exercise: (i) Give one sentence with a negation which is true relatives to M_1, (ii) Give one sentence with an implication which is true relative to M_1.

3.2. With variables

- **Interpreting individual variables**

  The above rules allow us to interpret constants and formulas. But, what about individual variables? The device that we use to interpret variables is called an **assignment function**, a function that specifies for each variable, how that variable is to be interpreted.

$$\text{For example: } g_1 = \begin{bmatrix} x & \rightarrow & \text{Andy} \\ y & \rightarrow & \text{Billy} \\ z & \rightarrow & \text{Cindy} \\ & \ldots & \end{bmatrix} \qquad g_2 = \begin{bmatrix} x & \rightarrow & \text{Billy} \\ y & \rightarrow & \text{Andy} \\ z & \rightarrow & \text{Cindy} \\ & \ldots & \end{bmatrix}$$

  Thus, to interpret expressions in a language that involves variables, we need both M and g. For an expression α, $[\![\alpha]\!]^{M,g}$ is called the *interpretation/denotation* of α relative to M and assignment function g.

(13) $[\![x]\!]^{M,g} = g(x)$, for example:
  a. $[\![x]\!]^{M,g_1} = g_1(x) = \text{Andy}$
  b. $[\![x]\!]^{M,g_2} = g_2(x) = \text{Billy}$

- **Interpreting quantification**

  Informally, $\forall x H(x)$ is true in a model M iff no matter which individual we assign to x, $H(x)$ is true. Likewise, $\forall x H(x)$ is true in a model M iff for some individual that we assign to x, $H(x)$ is true.

(14) $g[x \rightarrow k]$ is an assignment function that is exactly like g save that $g(x) = k$.

$$\text{For example: } g_1[x \rightarrow \text{Billy}] = \begin{bmatrix} x & \rightarrow & \text{Billy} \\ y & \rightarrow & \text{Billy} \\ z & \rightarrow & \text{Cindy} \\ & \ldots & \end{bmatrix}$$

(15) a. $[\![\exists x P(x)]\!]^{M,g} = 1$ iff $[\![P(x)]\!]^{M,g[x \rightarrow k]} = 1$ for some constant individual $k \in D$.
  b. $[\![\forall x P(x)]\!]^{M,g} = 1$ iff $[\![P(x)]\!]^{M,g[x \rightarrow k]} = 1$ for every constant individual $k \in D$.

Exercise: For each of the following wffs, determine its truth relative to M_1 and g_1.

(16) a. $H(x)$
  b. $\exists x H(x)$
  c. $\exists x \neg H(x)$
  d. $\forall x C(x)$
  e. $\forall x \exists y K(x, y)$
  f. $\forall x [H(x) \rightarrow C(x)]$

- **Semantics of predicate logic (Final!)**

(17) **Semantics of basic expressions**

a. If α is a non-logical constant, $[\![\alpha]\!]^{M,g} = I(\alpha)$

b. If α is an individual variable, $[\![\alpha]\!]^{M,g} = g(\alpha)$

(18) **Semantics of formulas**

a. *Equality*
If a and b are terms, $[\![a = b]\!]^{M,g} = 1$ iff $[\![a]\!]^{M,g} = [\![b]\!]^{M,g}$

b. *Predication*
If P is a one-place predicate, and a is a term:
$[\![P(a)]\!]^{M,g} = 1$ iff $[\![a]\!]^{M,g} \in [\![P]\!]^{M,g}$.
If P is an n-place predicate, and a_1, a_2, ..., a_n are terms:
$[\![P(a_1, a_2, ..., a_n)]\!]^{M,g} = 1$ iff $\langle [\![a_1]\!]^{M,g}, [\![a_2]\!]^{M,g}, ..., [\![a_n]\!]^{M,g} \rangle \in [\![P]\!]^{M,g}$.

c. *Negation and binary connectives*
If ϕ and ψ are formulas:
$[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$.
$[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$.
$[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$.
$[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$.
$[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$.

d. *Quantification*
If x is a variable and ϕ is a formula:
$[\![\exists x\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[x \rightarrow k]} = 1$ for some constant individual $k \in D$.
$[\![\forall x\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[x \rightarrow k]} = 1$ for every constant individual $k \in D$.

4. Translations between predicate logic and English

- **Predicates**

$P(t)$	1-place predicate	*run*, *happy*, *teacher*, *meet John*, *who arrived*$_{\text{relative clause}}$
$P(t_1, t_2)$	2-place predicate	*meet*, *taller than*, *come from*
$P(t_1, t_2, t_3)$	3-place predicate	*give*, *show*
$P(t_1, t_2, ..., t_n)$	n-place predicate	

– Translation of an atomic formula:

(19) a. Kitty meows
Key: $M(x)$: x meows; k: Kitty
Translation: $M(k)$

b. John introduced Andy to Billy.
Key: $I(x, y, z)$: x introduce y to z; j: John; a: Andy; b: Billy
Translation: $I(j, a, b)$

- **Quantifiers**

– Existential quantification:

(20) John admires some teacher.

a. $T(c) \wedge A(j, c)$ — Cindy is a teacher and John admires Cindy.

b. $\exists x[T(x) \wedge A(j, x)]$ — There is some x such that x is a teacher and John admires x.

– Universal quantification:

(21) Every teacher is friendly.

a. $T(p) \rightarrow F(p)$ — If Peter is a teacher, then he is friendly.

b. $T(b) \rightarrow F(b)$ — If Bill is a teacher, then he is friendly.

c. $\forall x[T(x) \rightarrow F(x)]$ — For every x, If x is a teacher, then x is friendly.

Discussion: The following wffs are not appropriate translations for (20) and (21). Why?

(22) a. $\exists x[T(x) \rightarrow A(j, x)]$

b. $\forall x[T(x) \wedge F(x)]$

Exercise: Translate the following formulas into good English sentences. (H: *be human*, L: *like*, m: *Mary*)

(23) a. $\exists x[H(x) \wedge L(x, m)]$

b. $\forall x[H(x) \rightarrow L(x, m)]$

c. $\exists x[H(x) \wedge \forall y[H(y) \rightarrow L(x, y)]]$

d. $\forall x[H(x) \wedge L(x, m) \rightarrow \exists y[H(y) \wedge L(x, y)]]$

Exercise: Translate the following sentences into predicate logic. (Ignore the details of tense and aspect.)

(24) a. Charles has a pen, but Elsa doesn't.

b. No one is going to visit John.

c. Every student enrolled in Semantics I is interested in some linguistic topic.

- **More about translations of quantificational expressions**

– Sometimes, the quantificational force of a sentence might not be explicitly expressed:

(25) a. A whale is a mammal. (Generic sentence)
$\forall x[W(x) \rightarrow M(x)]$

b. Students who are late are to be punished. (Bare plurals)
$\forall x[S(x) \wedge L(x) \rightarrow P(x)]$

– Scope ambiguity of quantifiers

(26) Every boy invited some girl.

a. Every boy is such that he invited a girl.
$\forall x[B(x) \rightarrow \exists y[G(x) \wedge I(x, y)]]$

b. There is a girl such that she is invited by every boy.
$\exists y[G(y) \wedge \forall x[B(x) \rightarrow I(x, y)]]$

– Interactions between negation and existential quantifiers

(27) a. John saw someone. $\exists x[H(x) \wedge S(j, x)]$

b. John didn't see someone.

c. John didn't see anyone.

d. John didn't see a person.