# Propositional logic

### Introduction

- **Translation and interpretation**
  - *Translation*: mapping from natural language expressions to meta-language expressions
  - *Interpretation*: mapping from meta-language expressions to their semantic values

(1) English sentence: This is Semantics I.
Translation in propositional logic: $p$
Interpretation: $[\![p]\!]^w = 1$

- A logic is a language, or a class of languages. Like any languages, a logic has vocabulary, syntax, and semantics. The grammar specifies the well-formed formulas (wffs) of the language. The semantics specifies the semantic value of every wff, given a valuation/model.

## 1. Vocabulary and syntax of propositional logic

- **Vocabulary**

(2) a. Propositional letters: $p, q, r, s, ..., \phi, \psi, ...$
b. Propositional connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
c. Parentheses: ( )
d. Connectives:

| Connectives | Compose proposition with connectives | | Translation |
|---|---|---|---|
| negation | $\neg p$ | (the negation of $p$) | it is not the case that $p$ |
| conjunction | $p \wedge q$ | (conjunction of $p$ and $q$) | $p$ and $q$ |
| disjunction | $p \vee q$ | (disjunction of $p$ and $q$) | $p$ or $q$ |
| (material) implication | $p \rightarrow q$ | (implication of $p$ and $q$) | if $p$, then $q$ |
| equivalence/bi-conditional | $p \leftrightarrow q$ | (equivalence of $p$ and $q$) | $p$ if and only if $q$ |

- **Well-formed formulas (wffs):** 'grammatically correct expressions' in propositional logic

(3) a. Every propositional letter is a wff.
b. If $p$ is a wff, then $\neg p$ is too.
c. If $p$ and $q$ are wffs, then so are $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, and $(p \leftrightarrow q)$.
d. Nothing else is a wff.

**Exercise:** Following the recursive rules above, identify whether each of the following strings is a wff.

(4) a. $(p)$
b. $\neg(\neg p)$
c. $\neg\neg p$
d. $(p \wedge q)$

- **Abbreviations** (to make formulas easier to read while do not carry any danger of ambiguity):

(5) Abbreviation rules

a. The outermost parentheses need not be explicitly mentioned.
$p \wedge q$ is short for $(p \wedge q)$

b. The conjunction and disjunction symbols apply to as little as possible.
$p \wedge q \rightarrow \neg r \vee s$ is short for $((p \wedge q) \rightarrow (\neg r \vee s))$

c. When one connective symbol is used repeatedly, grouping is to the right.[1]
$p \rightarrow q \rightarrow r$ is short for $p \rightarrow (q \rightarrow r)$

**Exercise:** Add parentheses to the following wff: $p \rightarrow q \rightarrow r \vee \neg s$

**Exercise:** Translate the following sentences into propositional logic.

(6) ex. John didn't hand in the exam.
$h$ = John handed in the exam. $\neg h$.

a. It is not the case that Guy comes if Peter or Harry comes.

b. John is not only stupid but also nasty.

c. Nobody laughed or applauded.

d. Charles and Elsa are brother and sister or nephew and niece.

## 2. Semantics of propositional logic

### 2.1. Truth values and truth functional

- In propositional logic, every wff/sentence $p$ has a *truth value* relative to an **assignment** of values to the propositional variables (also called a **valuation** $V$ or an **evaluation possible world** $w$). In a two-valued logic, every wff is assigned one of the following truth values: 1 (TRUE) or 0 (FALSE).

(7) Valuation as a total function from propositional variables to truth values[2]

$$w = \begin{bmatrix} p \rightarrow 1 \\ q \rightarrow 1 \\ r \rightarrow 0 \\ s \rightarrow 0 \\ \ldots \end{bmatrix}$$

- The truth value of a complex wff is computed based on (i) the truth-values of the connected simple sentences and (ii) the truth-functional properties of the connective(s).

(8) a. **Atomic formula**
If $p$ is an atomic formula, then $[\![p]\!]^w = w(p)$

b. **Negation**
$[\![\neg p]\!]^w = 1$ iff $[\![p]\!]^w = 0$.

c. **Binary connectives**
$[\![p \wedge q]\!]^w = 1$ iff $[\![p]\!]^w = 1$ and $[\![q]\!]^w = 1$.
$[\![p \vee q]\!]^w = 1$ iff $[\![p]\!]^w = 1$ or $[\![q]\!]^w = 1$.
$[\![p \rightarrow q]\!]^w = 1$ iff $[\![p]\!]^w = 0$ or $[\![q]\!]^w = 1$.
$[\![p \leftrightarrow q]\!]^w = 1$ iff $[\![p]\!]^w = [\![q]\!]^w$.

[1] I'd suggest not to use abbreviation rule (c); in other formal systems (such as $\lambda$-calculus), grouping is to the left.

[2] The valuation function $[\![\bullet]\!]^w$ can also be regarded as a characteristic function of a set consisting of propositions true in $w$.

### 2.2. Truth tables

- In a truth table, each row represents the truth values of the wffs relative to a different assignment of values to the propositional variables.

| $p$ | $q$ | $\neg p$ | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |

Table 1: Truth tables of propositional connectives

– More on disjunctions:

The disjunction $p \vee q$ is an *inclusive disjunction*. In natural languages, however, *p or q* is more likely to be interpreted exclusively: an *exclusive disjunction* is true iff exactly one of the disjuncts is true.

**Exercise:** Make a truth table for the exclusive disjunction.

– More on (material) implications:

Material implication $p \rightarrow q$ is defined truth-functionally. (Namely, $[\![p \rightarrow q]\!]^w$ is exclusively determined by $[\![p]\!]^w$ and $[\![q]\!]^w$.) The antecedent $p$ and the consequent $q$ do not necessarily have a causal relation.

(9) a. If Harvard is in Cambridge, then Cambridge has no university.
b. If Gennaro is a professor at Harvard, then Boston is in MA.

When the antecedent is false, the implication is vacuously true regardless of the value of the consequent.

(10) a. If 1+1 = 0, then Boston is in MA. TRUE
b. If 1+1 = 0, then Boston is not in MA. TRUE

### 2.3. Tautology, contradiction, contingency

- Terminologis:
  - *Tautology* ($\top$): Always true $p \vee \neg p$
  - *Contradiction* ($\bot$): Always false $p \wedge \neg p$
  - *Contingency:* Sometimes true and sometimes false $p$
- A selected list of tautologies (redundant brackets are omitted)

(11) Associative and commutative laws for $\wedge, \vee, \leftrightarrow$

(12) Distributive laws
a. $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$
b. $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$

(13) Negation
a. $\neg\neg p \leftrightarrow p$
b. $\neg(p \leftrightarrow q) \leftrightarrow ((p \wedge \neg q) \vee (\neg p \wedge q))$

(14) De Morgan's laws

a. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
b. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

(15) Eliminability of the material conditional[3]

a. $(p \to q) \leftrightarrow \neg(p \wedge \neg q)$
b. $(p \to q) \leftrightarrow (\neg p \vee q)$

(16) Others

a. $p \vee \neg p$ — Excluded middle
b. $\neg(p \wedge \neg p)$ — Contradiction
c. $(p \to q) \leftrightarrow (\neg q \to \neg p)$ — Contraposition
d. $((p \wedge q) \to r) \leftrightarrow (p \to (q \to r))$ — Exportation

- **Exercise:** Let $p$ be a tautology, $q$ a contradiction, and $r$ a contingency. Which of the following sentences are (i) tautological, (ii) contradictory, (iii) contingent, (iv) logically equivalent to $r$.

(17) a. $p \wedge r$
b. $p \vee r$
c. $q \wedge r$
d. $p \vee q$
e. $r \to q$

- Two ways of determining whether a wff is a tautology.
  - *Truth-table method:* We investigate every possible combination of truth-values for the simple sentences and then check the resulting truth-value of the complex expression.

Example: $\neg\neg p \to p$

| $p$ | $\neg p$ | $\neg\neg p$ | $\neg\neg p \to p$ |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |

Example: $p \to (q \to p)$

| $p$ | $q$ | $q \to p$ | $p \to (q \to p)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |

  - *Indirect reasoning:* Assume that the considered wff is false. If this assumption leads to a contradiction (i.e., impossible to be false), this wff is a tautology; otherwise it isn't.

Example: $[p \to (q \wedge r)] \to (p \to r)$

| | $[p$ | $\to$ | $(q$ | $\wedge$ | $r)]$ | $\to$ | $[p$ | $\to$ | $r]$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | 0 | | | |
| 2 | | 1 | | | | | | 0 | |
| 3 | | | | | | | 1 | | 0 |
| 4 | 1 | | | | 0 | | | | |
| 5 | 1 | | | 0 | | | | | |
| 6 | | 0 | | | | | | | |

- **Exercise:** Determine whether the following sentences are tautologous:

(18) a. $(p \to q) \to (q \to p)$
b. $p \vee (p \to q)$

[3] These tautologies show that material implication can be defined based on negation and conjunction/disjunction. More generally, any one of the three binary connectives ($\vee$, $\wedge$, $\to$) plus negation ($\neg$) is enough to capture the functionality of the other two.

### 2.4. Entailment and logical equilevance

- $p$ **entails** $q$ (written as $p \Rightarrow q$) iff $p \rightarrow q$ is a tautology.

  $p$ and $q$ are **(logically/semantically) equivalent** (written as $p \equiv q$) iff $p \leftrightarrow q$ is a tautology (or equivalently, $p \Rightarrow q$ and $q \Rightarrow p$)

  **Exercise:** Prove that $\neg p$ and $p \rightarrow \neg p$ are logically equivalent.

### 2.5. Propositions as sets of possible worlds

- A propositions can also be viewed as the set of possible worlds where this proposition is true (or equivalently, a function from a world to the truth value of this proposition in that world). Thus, we can use set-theoretic notations to represent propositions and relations of propositions.

(19) Atomic proposition
  a. $[\![p]\!] = \{w \mid [\![p]\!]^w = 1\}$
  b. $[\![\neg p]\!] = W - [\![p]\!]$

(20) Complex proposition
  a. $[\![p \wedge q]\!] = [\![p]\!] \cap [\![q]\!]$
  b. $[\![p \vee q]\!] = [\![p]\!] \cup [\![q]\!]$

(21) Relation
  a. $p$ entails $q$ iff $[\![p]\!] \subseteq [\![q]\!]$
  b. $p$ contradicts $q$ iff $[\![p]\!] \cap [\![q]\!] = \varnothing$

(22) Special propositions
  a. Contradiction: $\varnothing$
  b. Tautology: $W$

**Exercise**: Use set-theoretic notation to define an implication $p \rightarrow q$.