

Set theory, relations, and functions

1. Set theory

1.1. What is a set?

- A *set* is an abstract collection of distinct objects. Those objects are the *members/elements* of this set. In set theory, the concept 'set' is *extensional* (not *intensional*), namely, we don't bother about the ways in which the members of a set are chosen.

Exercise: Which of the following expressions correctly describe the features of set members?

- a. Distinct to each other
 - b. Ordered
 - c. Relevant to each other
 - d. The overall quantity is finite
- Some special sets
 - *Singleton set*: A set with exactly one member. E.g. $\{a\}, \{\{a, b\}\}$
 - *Empty/null set*: the set that has no member. Written as \emptyset or $\{\}$.
 - Convention of set-theoretic notations
 - (1) a. (Ordinary) sets: A, B, C, \dots
 - b. Members of a set: a, b, c, \dots, x, y, z
 - c. Sets of sets: $\mathcal{A}, \mathcal{B}, \mathcal{C}$
 - d. Cardinality of A : $|A|$ or $\#(A)$
 - e. membership relation: \in

Exercise: What is the cardinality of the set $\{a, a, b, b, c\}$?

Is $\{\emptyset\}$ a singleton set or an empty set? What about $\{\emptyset, \emptyset\}$?

- Ways to define a set
 - *list/roster notation*: listing all its members
 - (2) $E = \{4, 6, 8, \dots\}$Note: The order of the elements doesn't matter. Elements can be listed multiple times.
 - *predicate/abstraction notation*: stating a condition that is satisfied by all and only the elements of the set to be defined

Schema: $\{x \mid P(x)\}$, where x is a variable, and P is a predicate.

 - (3) a. $\{x \mid x \text{ is an even number greater than } 3\}$
Read as: "the set of all x such that x is an even number greater than 3"
 - b. $\{x \mid x \text{ is evenly divisible by } 2 \text{ and is greater than or equal to } 4\}$
 - *recursive rules*: using rules which generates/defines the members of this set.

- (4) a. $4 \in E$
 (the basis of recursion)
 b. If $x \in E$, then $x + 2 \in E$
 (to generate new elements from the elements defined before)
 c. Nothing else belongs to E .
 (to restrict the defined set to the elements generated by rules a and b)

1.2. Set-theoretic relations and operations

• Relations of sets

(5) a. **Identity/Equality**

Two sets are identical/equal iff they have exactly the same members.

$A = B$ iff for every x : $x \in A$ iff $x \in B$.

b. **Subsets**

i. A is a *subset* of B iff every element of A is an element of B .

$A \subseteq B$ iff for every x : if $x \in A$, then $x \in B$.

ii. A a *proper subset* of B iff A is a subset of B but not identical to B .

$A \subset B$ iff $A \subseteq B$ and $A \neq B$.

c. **Supersets**

i. A is a *superset* of B iff every element of B is an element of A (or say, B is a subset of A).

$A \supseteq B$ iff $B \subseteq A$

ii. A a *proper superset* of B iff B is a proper subset of A .

$A \subset B$ iff $B \subset A$

d. **Power sets**

i. Given a set A , the set of all the subsets of A is called the *power set* of A , written as $\mathcal{P}(A)$.

ii. \emptyset is a subset of every set.

Exercise: Let S be an arbitrary set, consider:

- a. is S a member of $\{S\}$?
 b. is $\{S\}$ a member of $\{S\}$?
 c. is $\{S\}$ a subset of $\{S\}$?
 d. what is the set whose only member is $\{S\}$?

• Operations on sets

(6) a. **Union**

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

(or equivalently: $\cup\{A, B\}$)

b. **Intersection**

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

(or equivalently: $\cap\{A, B\}$)

c. **Difference**

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

also called: the (*relative*) complement of B in A

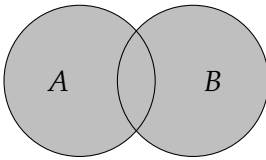
d. **(Absolute) Complement**

$A' = U - A$

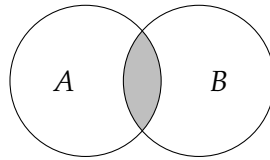
$= \{x \mid x \in U \text{ and } x \notin A\}$

Other notations: $\complement A$, A^c , or \overline{A}

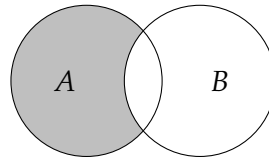
We can use Venn diagrams to illustrate these operations.



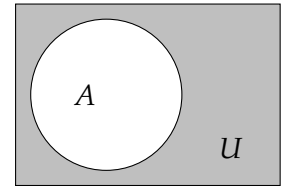
Union $A \cup B$



Intersection $A \cap B$



Difference $A - B$



Complement A'

Exercise: Simplify the following set-theoretic notations

- (7) a. $\{x \mid x = a\}$
 b. $\{x \mid x \neq x\}$
 c. $(A - B) - B'$

1.3. Applications of set theory in natural language semantics

- The following categories denote sets of individuals:

– common noun (*flower, cat*)

$$\llbracket \text{cat} \rrbracket = \{x \mid x \text{ is a cat}\}$$

– predicative adjective (*happy, gray*)

$$\llbracket \text{gray} \rrbracket = \{x \mid x \text{ is gray}\}$$

– VP and intransitive verb (*wears a hat, run, meow*)

$$\llbracket \text{wears a hat} \rrbracket = \{x \mid x \text{ wears a hat}\}$$

- Quantificational determiners (e.g., *every, some*) denote relations of sets

(8) a. Every cat meows.

$$\{x \mid x \text{ is a cat}\} \subseteq \{x \mid x \text{ meows}\}$$

b. Some cat meows.

$$\{x \mid x \text{ is a cat}\} \cap \{x \mid x \text{ meows}\} \neq \emptyset$$

Exercise: Use set-theoretic notations to represent the meaning of the following sentences.

- (9) a. Every gray cat meows.
 b. No cat barks.
 c. Two cats are meowing.

2. Relations

2.1. Ordered pairs and Cartesian products

- The elements of a set are not ordered. To describe functions and relations we will need the notion of an **ordered pair**, written as $\langle a, b \rangle$ (or (a, b)), where a is the first element of the pair and b is the second. Compare,

$$(10) \text{ If } a \neq b, \text{ then } \{a, b\} = \{b, a\}, \text{ but } \langle a, b \rangle \neq \langle b, a \rangle \\ \{a, a\} = \{a, a, a\}, \text{ but } \langle a, a \rangle \neq \langle a, a, a \rangle$$

- The **Cartesian product** of two sets A and B (written as $A \times B$) is the set of ordered pairs which take an element of A as the first member and an element of B as the second member.

$$(11) \ A \times B = \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$$

E.g. Let $A = \{1, 2\}$, $B = \{a, b\}$, then $A \times B = \{\langle 1, a \rangle, \langle 1, b \rangle, \langle 2, a \rangle, \langle 2, b \rangle\}$,
 $B \times A = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle\}$

2.2. Relations, domain, and range

- A *relation* is a set of ordered pairs. For example:

- Relations in math: $=, >, \neq, \dots$
- Relations in natural languages: the instructor of, the capital city of, ...

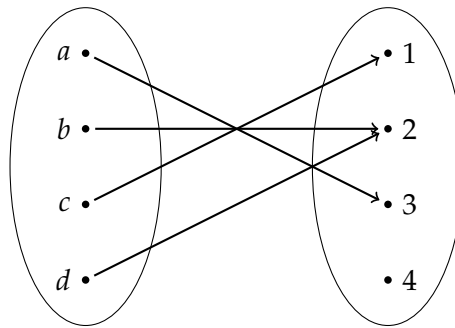
- (12) a. $\llbracket \text{the capital city of} \rrbracket = \{ \langle \text{USA, Washington} \rangle, \langle \text{China, Beijing} \rangle, \langle \text{France, Paris} \rangle, \dots \}$
 $= \{ \langle x, y \rangle \mid y \text{ is the capital city of } x \}$
- b. $\llbracket \text{invited} \rrbracket = \{ \langle \text{Andy, Billy} \rangle, \langle \text{Cindy, Danny} \rangle, \langle \text{Emily, Flori} \rangle, \dots \}$
 $= \{ \langle x, y \rangle \mid x \text{ invited } y \}$

- R is a *relation from* A *to* B iff R is a subset of the Cartesian product $A \times B$, written as $R \subseteq A \times B$.

R is a *relation in* A iff R is a subset of the Cartesian product $A \times A$, written as $R \subseteq A \times A$.

- (13) a. $\llbracket \text{the capital city of} \rrbracket \subseteq \{x \mid x \text{ is a country} \} \times \{y \mid y \text{ is a city} \}$
 b. $\llbracket \text{the mother of} \rrbracket \subseteq \{x \mid x \text{ is a human} \} \times \{y \mid y \text{ is a human} \}$

- We can use a *mapping diagram* to illustrate a relation:



- A and B are the **domain** and **range** of R respectively, iff $A \times B$ is the **smallest** Cartesian product of which R is a subset. For example:

- (14) Let $R = \{ \langle 1, a \rangle, \langle 1, b \rangle, \langle 2, b \rangle, \langle 3, b \rangle \}$, then: $\text{Dom}(R) = \{1, 2, 3\}$, $\text{Range}(R) = \{a, b\}$

Discussion: Why is that the following definitions are problematic? Provide some counterexamples.

- (15) a. A and B are the domain and range of R iff $R = A \times B$.
 b. A and B are the domain and range of R iff $R \subseteq A \times B$.

2.3. Properties of relations

• Reflexivity, symmetry, and transitivity

Given a set A and a relation R in A , ...

– R is *reflexive* iff for every x in A , $\langle x, x \rangle \in R$; otherwise R is *nonreflexive*.

If there is no pair of the form $\langle x, x \rangle$ in R , then R is *irreflexive*.

– R is *symmetric* iff for every xy in A , if $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$; otherwise R is *nonsymmetric*.

If there is no pair $\langle x, y \rangle$ in R such that the pair $\langle y, x \rangle$ is in R , then R is *asymmetric*.

– R is *transitive* iff for every xyz in A , if $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$; otherwise R is *nontransitive*.

If there are no pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R such that the pair $\langle x, z \rangle$ is in R , then R is *intransitive*.

Exercise: Identify the properties of the following relations w.r.t. reflexivity, symmetry, and transitivity. (Make the strongest possible statement. For example, call a relation 'irreflexive' instead of 'nonreflexive', if satisfied.)

- (16) a. = reflective, symmetric, transitive
b. \leq
c. \neq
d. \subseteq
- (17) a. is a sister of
b. is a mother of

- It is helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in *relational diagrams*. If x is related to y (namely, $\langle x, y \rangle \in R$), an arrow connects the corresponding points.

3. Functions

3.1. What is a function?

- A relation is a function iff each element in the domain is paired with **just one** element in the range.

$$(18) f(x) = x^2 \text{ for } x \in \mathbb{N}$$

In particular,

- a function f is a *from A (in)to B* (written as ' $f: A \rightarrow B$ ') iff $\text{Dom}(f) = A$ and $\text{Range}(f) \subseteq B$.
- a function f is a *from A onto B* iff $\text{Dom}(f) = A$ and $\text{Range}(f) = B$.

- We may specify functions with lists, tables, or words.

$$(19) \text{ a. } F = \{\langle a, b \rangle, \langle c, b \rangle, \langle d, e \rangle\}$$

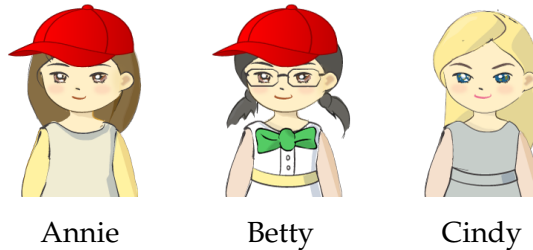
$$\text{b. } F = \begin{bmatrix} a \rightarrow b \\ c \rightarrow b \\ d \rightarrow e \end{bmatrix}$$

c. F is a function f with domain $\{a, b, c\}$ such that $f(a) = f(c) = b$ and $f(d) = e$.

Exercise: Define negation *it is not the case that* and conjunctive *and* as functions.

3.2. Characteristic function

- Recall that the following categories can be interpreted as sets of individuals: common nouns, predicative adjectives, intransitive verbs, VPs, ...



- (20) a. $\llbracket \text{wears a hat} \rrbracket = \{\text{Andy, Betty}\}$
 b. $\llbracket \text{wears a hat} \rrbracket = \{x \mid x \text{ wears a hat}\}$

Alternatively, the semantics of these expressions can also be modeled as functions from the set of individual entities to the set of truth values.

- (21) a. $\llbracket \text{wears a hat} \rrbracket = \{\langle \text{Andy}, 1 \rangle, \langle \text{Betty}, 1 \rangle, \langle \text{Cindy}, 0 \rangle\}$
 b. $\llbracket \text{wears a hat} \rrbracket = \begin{bmatrix} \text{Andy} \rightarrow 1 \\ \text{Betty} \rightarrow 1 \\ \text{Cindy} \rightarrow 0 \end{bmatrix}$
 c. $\llbracket \text{wears a hat} \rrbracket =$ the function f from individual entities to truth values such that $f(x) = 1$ if x wears a hat, and $f(x) = 0$ otherwise.

Characteristic function: a function whose range is the set of possible truth values $\{1, 0\}$.

Exercise: provide the characteristic function for wears a bowtie.

- Relations as functions into sets:** since a relation can be defined as a set of pairs, and any set has a characteristic function, relations can be equivalently conceptualized as functions from pairs to truth values.

- (22) a. $\llbracket \text{is next to} \rrbracket = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, b \rangle\}$
 b. $\llbracket \text{is next to} \rrbracket = \begin{bmatrix} \langle a, b \rangle \rightarrow 1 \\ \langle b, a \rangle \rightarrow 1 \\ \langle b, c \rangle \rightarrow 1 \\ \langle c, b \rangle \rightarrow 1 \\ \langle a, c \rangle \rightarrow 0 \\ \langle c, a \rangle \rightarrow 0 \end{bmatrix}$
 c. $\llbracket \text{is next to} \rrbracket =$ the function f from pairs of individual entities to truth values such that $f(x, y) = 1$ if x is next to y , and $f(x, y) = 0$ otherwise.